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Unitary highest weight representation of $U_q(su(1,1))$ when q is a root of unity

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Abstract. Unitary representation of $U_q(su(1,1))$ with q being roots of unity are studied. We construct unitary irreducible highest weight modules and find that the representations are discrete series owing to the unitarity. Moreover, it is revealed that each unitary irreducible highest weight module is equivalent to the tensor product of two modules. We show that one of them is just the unitary irreducible highest weight module of $su(1,1)$ and the other is the same as that of $U_q(su(2))$. It is also shown that the number of the latter modules is finite in our representation.

1. Introduction

Quantum groups have been playing important roles in recent developments of mathematical physics. They emerged initially in studying Yang–Baxter equations and quantum inverse scattering in statistical models [1, 2]. Quantum groups or quantum deformations of universal enveloping algebras provide rich algebraic structures for the Yang–Baxter equations in the exactly solvable models, knot theory and so on. This is the origin of integrability. Most remarkable developments have been made in its connection with two-dimensional solvable models, rational conformal field theory (RCFT) [3–10]. RCFT is characterized by a finite number of primary fields with respect to some chiral algebra. Well known examples are provided by the models of Virasoro minimal series [11], and unitary series [12] and Wess–Zumino–Witten (WZW) models whose chiral algebra is the Kac–Moody algebra [13]. The quantum group structure appears in these models through the quasi-triangularity of the Hopf algebra. In each model of RCFT, there is a parameter which characterizes the model. The parameter of a model and the deformation parameter of the quantum group which corresponds to the model are linked together by some relation. The central charge $c = 1 - 6/m(m+1)$ which is the parameter characterizing unitary minimal series are related with the deformation parameter q of $U_q(su(2))$ by the relation $q = \exp(2\pi i m/(m+1))$ [14]. On the other hand, $\widehat{su}(2)_k$ WZW model whose model-dependent parameter is the level k is connected to quantum group $U_q(su(2))$ by the relation $q = \exp(2\pi i/(k+2))$. The

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essential point is that these deformation parameters of *compact* quantum groups are roots of unity. Thus special interest for physicists lies in the quantum groups with q being a root of unity. Mathematically it is well-known that the representation theory for a quantum universal enveloping algebra when q is a root of unity is drastically different from the classical one [15].

In contrast to the powerful investigations of compact quantum groups, little attention has been paid to *non-compact* quantum groups. As for the non-compact quantum groups, unitary representations of $SU_q(1,1)$ when q is not a root of unity has been made in [16]. The authors of [17] have given a representation of $U_q(su(1,1))$ in terms of quasi-primary fields and discussed quantum Ward identities which are satisfied by correlation functions of the fields. However, as in the case of compact quantum groups, it would be expected that representations when q is a root of unity are essential for physics. In this paper, we will give an unitary highest weight representation of q-deformed universal enveloping algebra $U_q(su(1,1))$ when q is a root of unity.

Let us review here a highest weight representation of the classical Lie algebra $sl(2,\mathbf{R})\cong su(1,1)$ which is generated by L_0, $L_{\pm1}$. The representation is discrete series where L_0 eigenvalue of the highest weight state takes values $h=1,\frac{3}{2},\cdots$. Unitarity of the representation puts restriction on the value of h to be $h>\frac{1}{2}$ and the fact that h must be a half-integer comes from the single-valuedness of the actions of $SL(2,\mathbf{R})$ group rather than Lie algebra $sl(2,\mathbf{R})$ on a vector space of representation.

For quantum groups, however, we have only the universal enveloping algebra $U_q(su(1,1))$ in hand. Are the highest weight representations discrete series? To what extent are they unitary? The aim of this paper is to answer these questions. We will construct unitary highest weight module when q is a root of unity. It will be turned out that, as far as we require the norms of states to be positive and finite, the unitary irreducible highest weight modules should be characterized by two integers, that is, the representations are discrete series, although the values of highest weights are no longer half-integers. We will investigate the two-parameter structure of the unitary module and find that the module is isomorphic to a tensor product of two unitary modules. One of the modules is the module on which the classical Lie algebra $su(1,1)$ acts. The quantum group natures appear only in the other module. The remarkable point is that the latter module is finite dimensional, that is, the number of highest weight states is finite owing to the unitarity. Furthermore, each unitary irreducible highest weight module consists of finite number of states. It will be turned out that these modules are nothing but the unitary irreducible highest weight modules of $U_q(su(2))$! Thus the non-compact nature appears only in the former one.

This paper is organized as follows: in section 2, we present a brief introduction of $U_q(su(1,1))$ and its highest weight representation. In section 3, we go on to the construction of irreducible highest weight representation when q is a root of unity. Here we will see that there should be zero-norm in a highest weight module so that the norms of all the states in the module may be finite and the requirement of the appearance of the zero-norm states makes the highest weight representation discrete series. The highest weight module, however, is no longer irreducible because zero-norm states cause submodules. We construct an irreducible highest weight module by subtructing the submodules. Section 4 is devoted to the discussions of unitarity of the module. The meaning of the two-parameter structure which appeared in section 3 will become clear in section 5. We conclude in section 6.

2. $U_q(su(1,1))$ and its highest weight representation

Let us first summarize the q-deformation of the universal enveloping algebra of $su(1,1)$, $U_q(su(1,1))$, when the deformation parameter q is generic.

Definition 1. The algebra is defined by the following generators and relations;

$$\text{Generators: } \mathscr{L}_1, \mathscr{L}_{-1}, \mathscr{K}, \mathscr{K}^{-1} \tag{2.1}$$

$$\text{Relations: } [\mathscr{L}_1, \mathscr{L}_{-1}] = \frac{\mathscr{K}^2 - \mathscr{K}^{-2}}{q - q^{-1}} \tag{2.2}$$

$$\mathscr{L}_{\pm 1}\mathscr{K} = q^{\pm 1}\mathscr{K}\mathscr{L}_{\pm 1} \tag{2.3}$$

if we put $\mathscr{K} = q^{\mathscr{L}_0}$, then these relations (2.2) reduce to those of algebra $su(1,1)$; $[L_n, L_m] = (n-m)L_{n+m}$, $n, m = \pm 1, 0$, in the classical limit $q \to 1$. $U_q(su(1,1))$ is equipped with a structure of Hopf algebra, i.e. bi-algebra with anti-pode:

(i) co-product Δ: $U_q \to U_q \otimes U_q$ is an algebra homomorphism satisfying

$$\begin{aligned} \Delta(\mathscr{L}_{\pm 1}) &= \mathscr{L}_{\pm 1} \otimes \mathscr{K} + \mathscr{K}^{-1} \otimes \mathscr{L}_{\pm 1} \\ \Delta(\mathscr{K}) &= \mathscr{K} \otimes \mathscr{K} \end{aligned} \tag{2.4}$$

(ii) co-unit ε: $U_q \to C$

$$\varepsilon(\mathscr{L}_{\pm 1}) = 0 \qquad \varepsilon(\mathscr{K}) = 1 \tag{2.5}$$

(iii) anti-pode γ: $U_q \to U_q$ is an anti-homomorphism such that

$$\gamma(\mathscr{L}_{\pm 1}) = -q^{\mp 1}\mathscr{L}_{\pm 1} \qquad \gamma(\mathscr{K}) = \mathscr{K}^{-1}. \tag{2.6}$$

At this stage it is worth mentioning the difference between $U_q(su(2))$ and $U_q(su(1,1))$. The difference between these two lies only in the relation (2.2). the commutation relations between $\mathscr{K}$ and $\mathscr{L}_{\pm 1}$* of $U_q(su(2))$ are $\mathscr{L}_{\pm 1}\mathscr{K} = q^{\mp 1}\mathscr{K}\mathscr{L}_{\pm 1}$. The difference affects the structure of the representation space, namely, whether it is compact or not.

Let us study highest weight representation of $U_q(su(1,1))$. The highest weight representation is characterized by an $\mathscr{L}_0$ eigenvalue h, which is called a highest weight. The representation space V_h, we will refer to it as a highest weight module, is constructed by acting $\mathscr{L}_{-1}$ on the highest weight state $|h; 0\rangle$, which is defined by

$$\mathscr{L}_1|h; 0\rangle = 0 \tag{2.7}$$

$$\mathscr{K}|h; 0\rangle = q^h|h; 0\rangle. \tag{2.8}$$

As far as q is not a root of unity, the highest weight state is characterized by these equations completely. The module V_h consists of an infinite numbers of descendants $|h; r\rangle$ $(r \in N)$ which is defined by

$$|h; r\rangle \stackrel{\text{def}}{=} \frac{(\mathscr{L}_{-1})^r}{[r]!}|h; 0\rangle \tag{2.9}$$

where we use the notation of q-integer $[A]$

$$[A] = \frac{q^A - q^{-A}}{q - q^{-1}} \tag{2.10}$$

* For $U_q(su(2))$, notation $X^{\pm}$ and H are usually used instead of $\mathscr{L}_{\pm 1}$ and $\mathscr{L}_0$, respectively.

This convention for a q-integer is convenient for the later use in the sense that $[A]\in \boldsymbol{R}$, $\forall A\in \boldsymbol{R}$ when $|q|=1$. Using (2.9), the actions of $\mathcal{L}_1$, $\mathcal{L}_{-1}$ on a state $|h;r\rangle$ are easily obtained as follows:

$$\mathcal{L}_1|h;r\rangle=[2h+r-1]\,[h;r-1\rangle \tag{2.11}$$

$$\mathcal{L}_{-1}|h;r\rangle=[r+1]|h;r+1\rangle. \tag{2.12}$$

Therefore $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ act on the module V_h as a lowering operator and a raising operator, respectively. The norm of the state $|h;r\rangle$ is defined as an inner product $\||h;r\rangle\|^2=\langle h;r|h;r\rangle$ with respect to $\dagger$-conjugation:

$$\dagger: |h;r\rangle\to(|h;r\rangle)^\dagger=\langle h;r|\in V_h^\dagger.$$

The following proposition gives consistent $\dagger$-conjugations for non-compact quantum universal enveloping algebra $U_q(su(1,1))$.

Proposition 2. There exists consistent $\dagger$-conjugations with the relations (2.2) and (2.3) of $U_q(su(1,1))$ if and only if q is real or $|q|=1$. The corresponding $\dagger$-conjugations are given as follows:

$$(1)\ \mathcal{K}^\dagger=\mathcal{K},\ \mathcal{L}_{\pm1}^\dagger=\mathcal{L}_{\mp1} \qquad \text{when } q\in\boldsymbol{R}\backslash\{0\}$$

$$(2)\ \mathcal{K}^\dagger=\mathcal{K}^{-1},\ \mathcal{L}_{\pm1}^\dagger=\mathcal{L}_{\mp1} \qquad \text{when } |q|=1.$$

Using the definition of $\dagger$-conjugation and relations of $U_q(su(1,1))$, we obtain the norm of the state $|h;r\rangle$ as,

$$\||h;r\rangle\|^2=\begin{bmatrix}2h+r-1\\ r\end{bmatrix}_q \tag{2.13}$$

where we use the notation

$$\begin{bmatrix}n\\ r\end{bmatrix}_q=\frac{[n]!}{[n-r]![r]!}$$

and the normalization $\langle h;0|h;0\rangle=1$. When q is generic, i.e. $q\in\boldsymbol{R}\backslash\{0,\pm1\}$, all the states in the highest weight module V_h with values $h=1,\frac{3}{2},2,\ldots$ have positive definite norms and V_h has no submodule except for 0 and V_h itself, that is, V_h is a unitary irreducible highest weight module. This structure is the same as the classical one.

In the case that q is a root of unity, the situation is drastically different. It is worthwhile to give brief observations what happens in this case, especially q is a primitive mth root of unity. In order to define a universal enveloping algebra in the case $q^m=1$, further generators should be added to (2.1) and, therefore, the condition given in (2.7) is not sufficient to judge a state to be a highest weight state. Furthermore, the relation (2.8) is not sufficient to measure the weight of $|h;0\rangle$ because of periodicity of q^h under $h\to h+km$. Therefore, we will add another condition to (2.7) and change the relation (2.8) in order to specify a highest weight module. The structure of the highest weight module is also different. The norm of a state can be either negative or null. The finiteness of norms of all the states in a module requires existence of zero-norm states in the module. Some of these zero-norm states which are also highest weight states are called null states. That is to say, the null state is a highest weight state with zero-norm. Each null state causes a submodule and, therefore, the highest weight module is no longer irreducible. We will discuss details in the following sections.

3. Highest weight module with q being a root of unity

In this section, we try to construct an irreducible highest weight module when q is a root of unity without discussion of unitarity. Hereafter we set $q=e^{\pi i(n/m)}$ $(m>n)$, where m and n are positive integer and they are co-prime.

In this case, $q^m=(-)^n$ and then $[m]=0$. Because of this, the norm of the mth state $\||h;m\rangle\|^2$ diverges unless there exist an integer $\mu(\mu\leqslant m)$ satisfying

$$[2h+\mu-1]=0 \tag{3.1}$$

The equation (3.1) leads to the following proposition:

Proposition 3. The highest weight h is labelled by two integers μ and ν as follows;

$$h_{\mu\nu}=\frac{1}{2}\left(\frac{m}{n}\nu-\mu+1\right) \tag{3.2}$$

where $\mu=1,2,\ldots,m$ and $\nu\in N$. Then it satisfies the relation

$$[2h_{\mu\nu}+\mu-1]=0. \tag{3.3}$$

In the rest of this paper, we will impose the condition $\nu\geqslant(n/m)(\mu-1)$, since we are discussing highest weight representations, i.e., $h_{\mu\nu}>0$, rather than lowest weight representations, i.e. $h_{\mu\nu}<0$. Is there a possibility that the values of the highest weight $h_{\mu\nu}$ are the same for different values of μ and ν? In order to answer this question, the fact that the maximal value of μ is restricted to m is crucial. Indeed, if $1\leqslant\mu,\mu'\leqslant m$ and $\nu,\nu'\in N$, then the identification $h_{\mu\nu}=h_{\mu'\nu'}$ holds if and only if $\mu=\mu'$ and $\nu=\nu'$. Proposition 3 suggests that we need two pieces of information to specify the highest weight state completely.

Let us begin to construct the highest weight module $V_{\mu\nu}$ which is built on the highest weight state $|h_{\mu\nu};0\rangle$. First of all, as was suggested in the last paragraph of the previous section, we should redefine the highest weight module when $q=e^{\pi i n/m}$. It is easily seen that $\mathcal{L}_{-1}^m$ acts on V_h as a null operator, that is, $\mathcal{L}_{-1}^m|\vartheta\rangle=0$ for $\forall|\vartheta\rangle\in V_h$. Similarly, $\mathcal{L}_1^m$ is also null operator on V_h owing to the relation (3.1). However, $\mathcal{L}_{\pm1}^m/[m]!$ are not. Therefore we have to add $\mathcal{L}_{\pm1}^m/[m]!$ to the generators of $U_q(su(1,1))$. Furthermore, (3.2) indicates that we cannot measure the weight in terms of (2.8). Indeed, the ν dependence of the $\mathcal{K}$ eigenvalue $q^{h_{\mu\nu}}$ appears only through the sign factor $(-)^\nu$. On the other hand, the operator $\mathcal{L}_0$ is enough to measure the weight completely. Hence, we redefine the highest weight module as follows.

Definition 4. When $q=e^{\pi i(n/m)}$, the highest weight module $V_{\mu\nu}$ on the highest weight state $|h_{\mu\nu};0\rangle$ is

$$V_{\mu\nu}=\{|h_{\mu\nu};r\rangle|r=0,1,2,\ldots\}$$

such that

$$\mathcal{L}_1|h_{\mu\nu};0\rangle=\frac{\mathcal{L}_1^m}{[m]!}|h_{\mu\nu};0\rangle=0 \tag{3.4}$$

$$\mathcal{L}_0|h_{\mu\nu};0\rangle=h_{\mu\nu}|h_{\mu\nu};0\rangle. \tag{3.5}$$

In the following, we divide our discussions into two cases, (i) $1\leqslant\mu\leqslant m-1$ and $\mu=m$, because the modules in the case (i) have zero-norm states but those in case (ii) do not.

(i) $1 \leqslant \mu \leqslant m-1$

In this case, $V_{\mu\nu}$ has states with zero norms and, by definition, the first zero-norm state appears at the μth level.

$$\| |h_{\mu\nu};\mu\rangle \|^2 = \begin{bmatrix} 2h_{\mu\nu}+\mu-1 \\ \mu \end{bmatrix}_q = 0 \tag{3.6}$$

It is important to notice that the state $|h_{\mu\nu};\mu\rangle$ is a null state. Indeed, by acting $\mathscr{L}_1$ on the state, it vanishes owing to the relations (2.11) and (3.3). Moreover, $((\mathscr{L}_1)^m/[m]!)|h_{\mu\nu};\mu\rangle$ vanishes because $\mu<m$ that is to say, the corresponding state does not exist in $V_{\mu\nu}$. The weight of the state $|h_{\mu\nu};\mu\rangle$ is $h_{\mu\nu}+\mu=h_{-\mu\nu}$. Thus the null state $|h_{\mu\nu};\mu\rangle$ generates the submodule $V_{-\mu\nu}$ in the original module $V_{\mu\nu}$. In order to obtain an irreducible module, we have to subtract the submodule generated by the null state. This is not the whole story, however. Some formulas which are useful for the next arguments are

$$\begin{aligned} &[2h_{\mu,\nu+2kn}+\mu-1]=0 \qquad k\geqslant 0 \\ &h_{\mu,\nu+2kn}+\mu=h_{-\mu,\nu+2kn}=h_{\mu,\nu}+km+\mu \end{aligned} \tag{3.7}$$

$$\begin{aligned} &[2h_{-\mu,\nu+2kn}+(m-\mu)-1]=0 \qquad k\geqslant 0 \\ &h_{-\mu,\nu+2kn}+m-\mu=h_{\mu,\nu+2(k+1)n}=h_{\mu,\nu}+(k+1)m. \end{aligned} \tag{3.8}$$

The first equation in (3.7) guarantees that the state $|h_{\mu,\nu+2kn};\mu\rangle$ is the first null state in the submodule $V_{\mu,\nu+2kn}$, that is, it is annihilated by the actions of $\mathscr{L}_1$ and $\mathscr{L}_1^m/[m]!$. The second line in (3.7) indicates that the state $|h_{\mu,\nu+2kn};\mu\rangle$, whose weight is the same as that of the state at the $(km+\mu)$th level in $V_{\mu\nu}$, generates a new submodule $V_{-\mu,\nu+2kn}$ in $V_{\mu,\nu+2kn}$. Similarly (3.8) means that the new submodule $V_{\mu,\nu+2(k+1)n}$ is generated by the state $|h_{-\mu,\nu+2kn};m-\mu\rangle$, which is the first null state in the submodule $V_{-\mu,\nu+2kn}$, and that this state appears at the $(k+1)m$th level in the original module $V_{\mu,\nu}$. According to the above discussions, we obtain the embeddings of submodules in the original highest weight module:

$$V_{\mu,\nu}\to V_{-\mu,\nu}\to V_{\mu,\nu+2n}\to V_{-\mu,\nu+2n}\to\cdots\to V_{\mu,\nu+2kn}\to V_{-\mu,\nu+2kn}\to V_{\mu,\nu+2(k+1)n}\to\cdots$$

Hence the irreducible highest weight module over the highest weight state $|h_{\mu,\nu};0\rangle$ can be obtained by the correct subtraction as follows:

$$V^{\mathrm{irr}}_{\mu,\nu}=\bigoplus_{k=0}^{\infty} V^{(k)}_{\mu,\nu} \tag{3.9}$$

where $V^{(k)}_{\mu,\nu}=V_{\mu,\nu+2kn}-V_{-\mu,\nu+2kn}$. The irreducible module is depicted in figure 1, where each state in the module $V_{\mu,\nu}$ is represented by a rigid line or a dotted line. The irreducible module $V^{\mathrm{irr}}_{\mu,\nu}$ consists of only the rigid lines. Note that all subtracted states, corresponding to dotted lines, have zero-norm but they are not necessarily null states. The upward and downward arrows stand for the actions of $\mathscr{L}_1$ and $\mathscr{L}_{-1}$, respectively, and curved arrows correspond to the actions of $\mathscr{L}^m_{\pm 1}/[m]!$.

Now we have obtained the irreducible highest weight module $V^{\mathrm{irr}}_{\mu,\nu}$ with $1\leqslant\mu\leqslant m-1$. Finally we calculate the character of this representation.

$$\chi_{\mu,\nu}(x)=\sum_{k=0}^{\infty}\frac{x^{h_{\mu,\nu+2kn}}}{1-x}-\sum_{k=0}^{\infty}\frac{x^{h_{-\mu,\nu+2kn}}}{1-x}=\frac{x^{h_{\mu,\nu}}}{(1-x)}\frac{(1-x^{\mu})}{(1-x^m)} \tag{3.10}$$

(ii) $\mu=m$

In this case, the module $V_{m,\nu}$ does not contain states with zero-norm. The module is irreducible of itself and is the same as the classical ones except the fact that the states

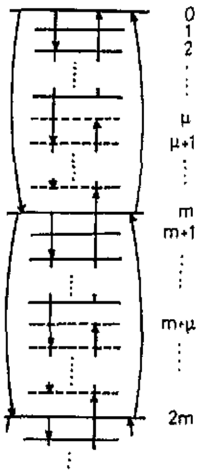

Figure 1. The irreducible highest weight module when $1 \leqslant \mu \leqslant m-1$. All states are described by rigid or dotted lines. Null states which must be subtracted are represented by dotted lines.

at the (km)th level $(k \geqslant 1)$ is generated by the action of $(\mathscr{L}_{-1})^m/[m]!$ on the state $|h_{\mu\nu}; (k-1)m\rangle$ instead of the action of $\mathscr{L}_{-1}$ on $|h_{\mu,\nu}; km-1\rangle$. In figure 2, the highest weight module $V_{\mu,\nu}$ with $\mu = m$ is described. The character in this case is the same as the classical one:

$$\chi_{m,\nu}(x) = \frac{x^{h_{m,\nu}}}{1-x} \tag{3.11}$$

In this section we have ignored discussions of positivity of the states, namely, unitarity of the module. This is detailed in the next section.

4. Discussion of unitarity

Now we are at the stage of studying the unitarity of the irreducible highest weight module obtained in the previous section. It will turn out that the values of the

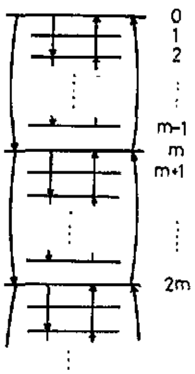

Figure 2. The irreducible highest weight module when $\mu = m$.

parameter μ and ν are more restricted. Our statement is summarized in the following theorem:

Theorem 5. Assume $q=e^{\pi i(n/m)}$ and let $\hat{m}$ be the maximal integer which does not exceed m/n. The cases in which unitary highest weight module exists are:
(i) If $m, n\in 2N-1$

$$\text{when } n\neq 1 \qquad \begin{cases}\mu, \nu\in 2N-1 & 1\leqslant\mu\leqslant\hat{m}+1\\ \text{or} \quad \mu=m & \nu\in 2N-1\end{cases}$$

$$\text{when } n=1 \qquad \mu, \nu\in 2N-1 \qquad 1\leqslant\mu\leqslant m.$$

(ii) If $m\in 2N$, $n\in 2N-1$

$$\text{when } n\neq 1 \qquad \begin{cases}\mu\in 2N & \nu\in 2N-1 & 1\leqslant\mu\leqslant\hat{m}+1\\ \text{or} & \mu=m & \nu\in 2N-1\end{cases}$$

$$\text{when } n=1 \qquad \mu\in 2N \qquad \nu\in 2N-1 \qquad 1\leqslant\mu\leqslant m.$$

(iii) If $m\in 2N-1$, $n\in 2N$

$$\mu=m \qquad \nu\in 2N.$$

The remainder of this section is devoted to the proof of this theorem.

The following consideration makes the discussion of the signs of all the states in a module easy. Let ε and $\varepsilon(r)$ be the signs of the states at the mth level and at the rth $(1\leqslant r\leqslant\mu-1)$ level, respectively. Then the sign of the $(mk+r)$th $(k\geqslant 1)$ level is $\varepsilon^k\times\varepsilon(r)$. Therefore, we have only to check the signs ε and $\varepsilon(r)$. The norm of the state at the mth level can easily be calculated as

$$\||h_{\mu,\nu}; m\rangle\|^2=((-)^{\nu+1})^{m-1}((-)^n)^{m-\mu}\left\{(-)^{\nu+n}\frac{\nu}{n}\right\} \tag{4.1}$$

where we used the relation $[m+x]=(-)^n[x]$ and $[-x]=-[x]$. Careful calculation of the combination $[2h_{\mu,\nu}+\mu-1]/[m]$, in which both the denominator and the numerator are zero, gives the last factor $\{\cdots\}$ in (4.1). From (4.1) we obtain the following lemma:

Lemma 6. The norm $\||h_{\mu,\nu}; m\rangle\|^2$ is positive if m, n and μ, ν satisfy
(a) if $m, n\in 2N-1$, then $\mu+\nu\in 2N$.
(b) if $m\in 2N$ and $n\in 2N-1$, then $\mu\in 2N$ and $\forall\nu\in N$.
(c) if $m\in 2N-1$ and $n\in 2N$, then $\nu\in 2N$ and $\forall\mu\in N$.

According to the discussion given above (4.1), the next task is to examine the signs of all the states lying between the highest weight state and the state at the $(\mu-1)$th level. The norm of the state $|h_{\mu,\nu}; r\rangle$ $(1\leqslant r\leqslant\mu-1)$ is given by

$$\||h_{\mu,\nu}; r\rangle\|^2=\begin{bmatrix}2h_{\mu,\nu}+r-1\\ r\end{bmatrix}_q=\prod_{j=1}^{r}\frac{\left[\frac{m}{n}\nu-\mu+j\right]}{[j]}=((-)^{\nu+1})^r\prod_{j=1}^{r}\frac{[\mu-j]}{[j]} \tag{4.2}$$

When $q=e^{\pi i(n/m)}$, the q-integer $[x]$ is written as

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}}=\frac{\sin(\pi xn/m)}{\sin(\pi xn/m)} \tag{4.3}$$

In the following, we will consider two cases (A) $n\neq 1$ and (B) $n=1$, separately.

(A) $n\neq 1$

Taking into account that the denominator in (4.3) is positive, the sign of $[x]$ is classified as follows:

$$\text{(a)}\ [x]>0\ \text{for}\ \left\langle\frac{2km}{n}\right\rangle+1\leqslant x\leqslant\left\langle\frac{(2k+1)m}{n}\right\rangle \qquad k\geqslant 0$$

$$\text{(b)}\ [x]<0\ \text{for}\ \left\langle\frac{(2k-1)m}{n}\right\rangle+1\leqslant x\leqslant\left\langle\frac{2km}{n}\right\rangle \qquad k\geqslant 1$$

where we use the symbol $\langle\xi\rangle$ as the maximal integer which does not exceed ξ. In particular, $\hat{m}\stackrel{\text{def}}{=}\langle m/n\rangle$. It should be noticed that

$$\left\langle\frac{(k+1)m}{n}\right\rangle-\left\langle\frac{km}{n}\right\rangle=\hat{m}\ \text{or}\ \hat{m}+1. \tag{4.4}$$

Equation (4.4) allows one to give the following classification (1)–(3) according to the value of $\mu(1\leqslant\mu\leqslant m-1)$:

(1) $\mu-1\leqslant\hat{m}$

In this case, all q-integers both in the numerator and in the denominator on the last term of (4.2) are positive, then the sign of the state $|h_{\mu,\nu;r}\rangle$, denoted as $\varepsilon(h_{\mu,\nu};r)$, is

$$\varepsilon(h_{\mu,\nu};r)=((-)^{\nu+1})^r \tag{4.5}$$

Therefore the sign $\varepsilon(h_{\mu,\nu};r)$ is positive for arbitrary r if and only if ν is odd.

(2) $\hat{m}+1\leqslant\mu=1\leqslant\left\langle\frac{2m}{n}\right\rangle$

In this case, three cases occur according to the values of r and μ.

(a) $r\leqslant\mu-1-\hat{m}$

We have to consider two cases, $r\leqslant\hat{m}$ and $r=\hat{m}+1$. When $r\leqslant\hat{m}$, negative signs come from all the q-integers in the numerator, then

$$\varepsilon(h_{\mu,\nu};r)=((-)^{\nu+1})^r(-)^r=((-)^\nu)^r. \tag{4.6}$$

In the case $r=\hat{m}+1$, which occurs only when $\mu-1=\langle 2m/n\rangle$ with $\langle 2m/n\rangle=2\hat{m}+1$, we have an additional minus sign coming from $[r]=[\hat{m}+1]$ in the denominator. Then

$$\varepsilon(h_{\mu,\nu};\hat{m}+1)=-((-)^\nu)^{\hat{m}+1}. \tag{4.7}$$

(b) $\mu-\hat{m}\leqslant r\leqslant\hat{m}$

The q-integers $[\mu-1], [\mu-2], \ldots, [\hat{m}+1]$ in the numerator provide negative signs, then

$$\varepsilon(h_{\mu,\nu};r)=((-)^{\nu+1})^r(-)^{\mu-\hat{m}-1}. \tag{4.8}$$

(c) $\hat{m}+1\leqslant r\leqslant\mu-1$

We have more negative signs from the q-integers $[\hat{m}+1], [\hat{m}+2], \ldots, [r]$ in the denominator in addition to the case (b), then

$$\begin{aligned}\varepsilon(h_{\mu,\nu};r)&=((-)^{\nu+1})^r(-)^{\mu-\hat{m}-1}(-)^{r-\hat{m}}\\&=((-)^\nu)^r(-)^{\mu-1}\end{aligned} \tag{4.9}$$

Putting these cases (a)–(c) together, we have to conclude that there are no numbers for μ and ν such that all the states $|h_{\mu,\nu}; r\rangle$ $(0 \leqslant r \leqslant \mu-1)$ have positive norms.

(3) $\langle km/n\rangle + 1 \leqslant \mu - 1 \leqslant \langle (k+1)m/n\rangle$ for $k \geqslant 2$
The situations for these cases are the same as the case (2), that is, we cannot find any numbers for μ and ν which give positive $\varepsilon(h_{\mu,\nu}; r)$, $\forall r$, $0 \leqslant r \leqslant \mu - 1$. Let us, first, examine the case $k \in 2N+1$. Assume that $y \overset{\text{def}}{=} \mu - 1 - \langle km/n\rangle$.

(a) $0 \leqslant r \leqslant y$
In this case, as in the case (2a), we have to consider two possibilities, i.e. $r \leqslant \hat{m}$ and $r = \hat{m}+1$. If $r \leqslant \hat{m}$, all the q-integers in the numerator are negative whereas ones in the denominator are positive, then

$$\varepsilon(h_{\mu,\nu}; r) = ((-)^{\nu+1})^r(-)^r = ((-)^\nu)^r. \tag{4.10}$$

When $r = \hat{m}+1$, which occurs only when $\mu - 1 = \langle (k+1)m/n\rangle$ with $\langle (k+1)m/n\rangle - \langle km/n\rangle = \hat{m}+1$, an additional minus sign arises:

$$\varepsilon(h_{\mu,\nu}; \hat{m}+1) = -((-)^\nu)^{\hat{m}+1}. \tag{4.11}$$

(b) $y+1 \leqslant r \leqslant \hat{m}$
Negative signs come from q-integers in the numerator $[\mu-1], [\mu-2], \ldots, [\langle km/n\rangle + 1]$ and, the number of minus signs is y, then

$$\varepsilon(h_{\mu,\nu}; r) = ((-)^{\nu+1})^r(-)^y. \tag{4.12}$$

We can deduce only from these two examinations that it is impossible to find μ and ν such that $\varepsilon(h_{\mu,\nu}; r) = +1$, $\forall r$.

When k is even, the same situations happen. From the above information (1)–(3), we can conclude that, when $1 \leqslant \mu \leqslant m-1$, all the states $|h_{\mu,\nu}; r\rangle$ are positive definite if and only if $1 \leqslant \mu \leqslant \hat{m}+1$ and ν is odd.

(4) $\mu = m$
In this case, the norm of the state $|h_{\mu,\nu}; r\rangle$ is given by

$$\begin{aligned} \| |h_{\mu,\nu}; r\rangle \|^2 &= ((-)^{\nu+1})^r \prod_{j=1}^{r} \frac{[m-j]}{[j]} \\ &= ((-)^{\nu+n})^r. \end{aligned} \tag{4.13}$$

Therefore, the condition that all the states $|h_{\mu,\nu}; r\rangle$ are positive definite is that

$$n + \nu \in 2N \tag{4.14}$$

(B) $n = 1$
Noticing that all q-integers $[x]$ for $x \leqslant m-1$ are positive, we can easily see that all the states $|h_{\mu,\nu}; r\rangle$ $(1 \leqslant r \leqslant \mu - 1)$ have positive norm if ν is odd. Moreover, (4.13) holds also for the case $n = 1$, then all the states $|h_{\mu,\nu}; r\rangle$ have positive norms when ν is odd.

Putting together the above investigations (A.1)–(A.4) and (B) taking lemma 6 into account, we have reached theorem 5.

Now we have obtained all unitary irreducible highest weight modules for non-compact $U_q(su(1,1))$-module, V_{mn}^{UCR} is obtained as a direct sum of the unitary irreducible highest weight modules as follows:

(A) $n \neq 1$

(1) $(m, n) \in (2N-1, 2N-1)$

$$V^{\mathrm{UCR}}_{m,n} = \left(\bigoplus_{\substack{\mu \in 2N-1 \\ 1 \leq \mu \leq \hat{m}+1}} \bigoplus_{\nu \in 2N-1} V^{\mathrm{irr}}_{\mu,\nu} \right) \oplus \left(\bigoplus_{n \leq \nu \in 2N-1} V^{\mathrm{irr}}_{m,\nu} \right) \tag{4.15}$$

(2) $(m, n) \in (2N, 2N-1)$

$$V^{\mathrm{UCR}}_{m,n} = \left(\bigoplus_{\substack{\mu \in 2N \\ 1 \leq \mu \leq \hat{m}+1}} \bigoplus_{\nu \in 2N-1} V^{\mathrm{irr}}_{\mu,\nu} \right) \oplus \left(\bigoplus_{n \leq \nu \in 2N-1} V^{\mathrm{irr}}_{m,\nu} \right) \tag{4.16}$$

(3) $(m, n) \in (2N-1, 2N)$

$$V^{\mathrm{UCR}}_{m,n} = \bigoplus_{n \leq \nu \in 2N} V^{\mathrm{irr}}_{m,\nu} \tag{4.17}$$

(b) $n = 1$

(1) $m \in 2N-1$

$$V^{\mathrm{UCR}}_{m,1} = \bigoplus_{\substack{\mu \in 2N \\ 1 \leq \mu \leq m}} \bigoplus_{\nu \in 2N-1} V^{\mathrm{irr}}_{\theta,\nu} \tag{4.18}$$

(2) $m \in 2N$

$$V^{\mathrm{UCR}}_{m,1} = \bigoplus_{\substack{\mu \in 2N \\ 1 \leq \mu \leq m}} \bigoplus_{\nu \in 2N-1} V^{\mathrm{irr}}_{\mu,\nu} \tag{4.19}$$

5. Two-parameter structure of the module $V^{\mathrm{UCR}}_{m,n}$

We have obtained the unitary completely reducible $U_q(su(1,1))$-module when q is a root of unity. However, there remains the question why we need two parameters μ and ν in order to specify a highest weight state. We have introduced the parameter μ in order to designate the level at which the first null state appears. On the other hand, we do not have a transparent meaning of the parameter ν. As a final discussion, we make the two-parameter structure of the highest weight module clear. It turns out that the module $V^{\mathrm{UCR}}_{m,n}$ can be split into two unitary modules, one of them parametrized by μ and the other by ν.

First of all, we should notice the following feature; on the module $\mathcal{V}^{\mathrm{UCR}} = V^{\mathrm{UCR}}_{m,n}/Z_2$, the universal enveloping algebra $U_q(su(1,1))$ can be written as a tensor-product of two algebra.

$$U_q(su(1,1)) = \tilde{U}_q \otimes U(su(1,1)) \tag{5.1}$$

where $\tilde{U}_q$ is generated by $\mathscr{L}_{\pm 1}$, $\mathscr{H}$ with the relations (2.2) (2.3) and $U(su(1,1))$ is a classical associative algebra whose generators are

$$\mathscr{G}_{\pm 1} \stackrel{\mathrm{def}}{=} \frac{\mathscr{L}^m_{\pm 1}}{[m]!}, \qquad \mathscr{G}_0 \stackrel{\mathrm{def}}{=} \frac{1}{2} \begin{bmatrix} 2\mathscr{L}_0 + m - 1 \\ m \end{bmatrix}_q \tag{5.2}$$

satisfying the relations

$$[\mathscr{G}_n, \mathscr{G}_m] = (n-m)\mathscr{G}_{n+m} \qquad n = 0, \pm 1 \tag{5.3}$$

and being equipped with a classical Hopf algebra. That is, on the module $\mathcal{V}^{\mathrm{UCR}}$, we can obtain the following homomorphisms:

$$\Delta(\mathcal{G}_n)=\mathcal{G}_n\otimes 1+1\otimes\mathcal{G}_n$$

$$\varepsilon(\mathcal{G}_n)=0$$

$$\gamma(\mathcal{G}_n)=-\mathcal{G}_n .$$

The reason why we deal with $\mathcal{V}^{\mathrm{UCR}}$ which is $V_{m,n}^{\mathrm{UCR}}$ modulo Z_2 is to remove the double-valuedness which arises from $q^m=(-)^n$. We can prove (5.1) by calculating the commutation relations among these generators, $\mathcal{L}_{\pm 1}$, $\mathcal{K}$ and $\mathcal{G}_{\pm 1}$, $\mathcal{G}_0$, restricted to $\mathcal{V}^{\mathrm{UCR}}$. Equation (5.1) derives the following theorem which states the structure of the unitary completely reducible $U_q(su(1,1))$ module $\mathcal{V}^{\mathrm{UCR}}$.

Theorem 7. (1) $\mathcal{V}^{\mathrm{UCR}}$ is isomorphic to a tensor product of two vector spaces as follows (this structure was first derived by Lusztig for the finite dimensional representation of $U_q(sl(2,C))$ [15]):

$$\mathcal{V}^{\mathrm{UCR}}\cong\hat{V}\otimes V^{\mathrm{cl}} \tag{5.4}$$

where

$$\hat{V}=\bigoplus_{\{\mu\}}\hat{V}_\mu\oplus\hat{V}_m \qquad V^{\mathrm{cl}}=\bigoplus_{\{\nu\}}V_\nu^{\mathrm{cl}}$$

$\tilde{U}_q$ and $su(1,1)$ act on $\mathcal{V}^{\mathrm{UCR}}$ as $\tilde{U}_q\otimes 1$ and $1\otimes su(1,1)$, respectively. The vector space $\tilde{V}_\mu$ (resp. V_ν^{cl}) is a unitary irreducible highest weight module whose highest weight depends only on μ (resp. ν). The parameters μ and ν with which the completely reducible modules are constructed are given according to theorem 5.
(2) Each unitary irreducible highest weight module V_ν^{cl} is equivalent to the unitary irreducible highest weight representation of the classical Lie algebra $su(1,1)$. Therefore, each irreducible highest weight module V_ν^{cl} is infinite dimensional.
(3) Each unitary highest weight module $\tilde{V}_\mu$ is equivalent to the unitary irreducible highest weight representation of $U_q(su(2))$ [18].
(4) The unitary irreducible highest weight modules $\tilde{V}_\mu$ and $\tilde{V}_m$ are of finite dimension

$$\dim\tilde{V}_\mu=\mu \qquad \dim\tilde{V}_m=m.$$

(5) The unitary completely reducible $\tilde{U}_q$-module $\tilde{V}$ is finite-dimensional, that is to say, the number of highest weight states in $\tilde{V}$ is finite. In fact, it is obtained from (4.15)–(4.19) by

$$\#\{\mu\}=\begin{cases}\langle m\rangle+2 & \text{when } n\neq 1\\ m & \text{when } n=1.\end{cases}$$

In what follows, we will investigate the modules $\tilde{V}$ and V^{cl} and the actions of $\tilde{U}_q$ and

$su(1,1)$ in detail. To this end, it is convenient to express the state $|h_{\mu,\nu}; km+r\rangle$ in terms of $\mathcal{L}_{-1}$ and $\mathcal{G}_{-1}$. Using the relation

$$\frac{1}{[km+r]!}=\{(-)^n\}^{1/2\,k(k-1)m+kr}\frac{1}{[r]!}\frac{1}{k!}\left(\frac{1}{[m]!}\right)^k \tag{5.5}$$

we can rewrite the state $|h_{\mu,\nu}; km+r\rangle$ as

$$h_{\mu,\nu}; km+r\rangle=\{(-)^n\}^{1/2\,k(k-1)m+kr}\frac{\mathcal{L}^r_{-1}}{[r]!}\frac{\mathcal{G}^k_{-1}}{k!}|h_{\mu,\nu}; 0\rangle. \tag{5.6}$$

Therefore, on the module $\mathcal{V}^{\mathrm{UCR}}$, i.e. $V^{\mathrm{UCR}}_{m,n}$ modulo $\mathbf{Z}_2$, we can write

$$|h_{\mu,\nu};\ km+r\rangle=|\tilde{h}; -\tilde{h}+r\rangle\otimes|\tilde{h}_c; k\rangle \tag{5.7}$$

where

$$|\tilde{h}; -\tilde{h}+r\rangle \overset{\mathrm{def}}{=} \frac{\mathcal{L}^r_{-1}}{[r]!}|\,\tilde{h}; -\tilde{h}\rangle \tag{5.8}$$

$$|h_c; k\rangle \overset{\mathrm{def}}{=} \frac{\mathcal{G}^k_{-1}}{[k]!}|\,h_c; 0\rangle \tag{5.9}$$

We will define the states $|h_c; 0\rangle^{\mathrm{cl}}$ and $|\tilde{h}; -\tilde{h}\rangle$ later.

Let us begin with the investigations of V^{cl} on which $\mathcal{G}_n$, $n=0, \pm 1$ act. The highest weight state $|h_c; 0\rangle^{\mathrm{cl}}$ is defined by the relations $\mathcal{G}_0|h_c; 0\rangle^{\mathrm{cl}}=h_c|h_c; 0\rangle^{\mathrm{cl}}$ and $\mathcal{G}_1|h_c; 0\rangle=0$ and its weight h_c is defined as a $\mathcal{G}_0$ eigenvalue of the original highest weight state $|h_{\mu,\nu}; 0\rangle$. Therefore, h_c is the just half of the norm of the mth state $|h_{\mu,\nu}; 0\rangle$

$$h_c=\frac{1}{2}\frac{\nu}{n}. \tag{5.10}$$

The highest weight module V^{cl}_ν consists of the states $|h_c; k\rangle^{\mathrm{cl}}$ defined by (5.9),

$$V^{\mathrm{cl}}_\nu=\left\{\left|\frac{1}{2}\frac{\nu}{n}; k\right\rangle|k=0, 1, 2, \ldots\right\}.$$

The action of $\mathcal{G}_{\pm 1}$, $\mathcal{G}_0$ are obtained as

$$\begin{aligned}\mathcal{G}_{-1}|h_c; k\rangle^{\mathrm{cl}}&=(k+1)|h_c; k+1\rangle^{\mathrm{cl}}\\ \mathcal{G}_1|h_c; k\rangle^{\mathrm{cl}}&=(2h_c+k-1)|h_c; k-1\rangle^{\mathrm{cl}}\\ \mathcal{G}_0|h_c; k\rangle^{\mathrm{cl}}&=(h_c+k)|h_c; k\rangle^{\mathrm{cl}}.\end{aligned} \tag{5.11}$$

Note that the module V^{cl}_ν is unitary and irreducible as far as the Lie algebra $su(1,1)$ is concerned and is non-compact space. Now, the meaning of the parameter ν becomes clear; it is related to the highest weight of the module V^{cl}_ν by the equation (5.10). The completely reducible su(1,1)-module V^{cl} is obtained by a direct sum of V^{cl}_ν with respect to ν:

$$V^{\mathrm{cl}}=\bigoplus_\nu V^{\mathrm{cl}}_\nu$$

where the allowed values for ν are given in theorem 5. We next investigate $\tilde{U}_q$-module $\tilde{V}$. The state $|\tilde{h}; -\tilde{h}\rangle$ introduced in (5.8), which we call the 'lowest' weight state, is defined by the relation $\mathcal{K}|\tilde{h}; -\tilde{h}\rangle=q^{-\tilde{h}}\,|\tilde{h}; -\tilde{h}\rangle$ and $\mathcal{L}_1|\tilde{h}; -\tilde{h}\rangle=0$. $\tilde{h}$ is defined by means of the relation $q^{h_{\mu,\nu}}=q^{-1/2\,i\pi\nu}q^{\tilde{h}}_c$ that is

$$\tilde{h}=\tfrac{1}{2}(\mu-1). \tag{5.12}$$

We see from (5.10) and (5.12) that the original highest weight $h_{\mu,\nu}$ is written as $h_{\mu,\nu}=h_c m-\tilde{h}$. The lowest weight module $\tilde{V}_\mu$ consists of the states $|\tilde{h};-\tilde{h}+r\rangle$ which are defined by (5.8):

$$\tilde{V}_\mu=\{|\tilde{h};-\tilde{h}+r\rangle|r=0,1,\ldots,2\tilde{h}\}.$$

The actions of the generators of $\tilde{U}_q$ on a state $|\tilde{h};-\tilde{h}+r\rangle\in\tilde{V}_\mu$ are obtained as follows

$$\mathcal{L}_{-1}|\tilde{h};-\tilde{h}+r\rangle=[r+1]|\tilde{h};-\tilde{h}+r+1\rangle \tag{5.13}$$

$$\mathcal{L}_1|\tilde{h};-\tilde{h}+r\rangle=-[2\tilde{h}-r+1]|\tilde{h};-\tilde{h}+r-1\rangle \tag{5.14}$$

$$\mathcal{K}|\tilde{h};-\tilde{h}+r\rangle=q^{-\tilde{h}+r}|\tilde{h};-\tilde{h}+r\rangle. \tag{5.15}$$

The extra minus sign in the right-hand side of (5.14) indicates that we should impose $\mathcal{L}_{-1}^\dagger=-\mathcal{L}_1$ on the †-conjugation rule in order to obtain positive norms $\||\tilde{h};-\tilde{h}+r\rangle\|^2$. This †-conjugation rule and the relations (2.2) and (2.3) guarantee that $\tilde{V}_\mu$ is just the unitary irreducible $U_q(su(2))$-module. The dimension of $\tilde{V}_\mu$ is

$$\dim\tilde{V}_\mu=2\tilde{h}+1=\mu. \tag{5.16}$$

The completely reducible $\tilde{U}_q$-module $\tilde{V}$ is described as a direct sum of the highest weight module $\tilde{V}_\mu$ with respect to the parameter μ.

$$\tilde{V}=\bigoplus_\mu\tilde{V}_\mu. \tag{5.17}$$

The allowed values for μ are given in theorem 5. It should be noticed that $\tilde{V}$ is of finite dimension, that is to say, the number of the lowest weight states is finite. Remember that this finiteness originates in the unitarity of the highest weight module $V_{\mu,\nu}$.

Finally we would like to stress that the non-compct nature of $\mathcal{V}_{m,n}^{\mathrm{UCR}}$ stems only from the module V^{cl} which is equivalent to the unitary highest weight representation of the classical Lie algebra $su(1,1)$ and the quantum group nature appears only in the finite dimensional module $\tilde{V}$.

6. Conclusion

In this paper, we have investigated the structure of the highest weight representation of $U_q(su(1,1))$ when q is a root of unity. The requirement of the unitarity claims the existence of zero-norm states in a highest weight module. Without zero-norm states, we have states whose norm diverge. This is the point where the difference between compact quantum groups and non-compact quantum groups appears. For compact quantum groups, there are the highest weight modules in which no zero-norm states exist [19]. We saw that the requirement of the existence of zero-norm states makes the representations of $U_q(su(1,1))$ discrete series. More precisely, we need two-parameters in order to specify a highest weight module. The two-parameter structure is meaningful. In fact, the $U_q(su(1,1))$-module is isomorphic to a tensor product of two modules V^{cl} and $\tilde{V}$. One of them, V^{cl}, is the unitary completely reducible module on which the classical Lie algebra $su(1,1)$ acts. This is a direct sum of unitary irreducible highest weight modules whose highest weights are specified by one of these parameters. Each highest weight module is equivalent to the unitary irreducible highest weight module of $su(1,1)$ and, therefore, is of infinite dimension. The other module, $\tilde{V}$ consists of a *finite* number of unitary irreducible highest weight modules whose highest weights are specified by the other parameter. Furthermore, each of these unitary irreducible modules is equivalent to the unitary irreducible highest weight module of $U_q(su(2))$ and, therefore, is finite-dimensional. Thus, the non-compact nature of unitary $U_q(su(1,1))$-module inherit only from the former V^{cl}, and the quantum group nature appears only in the latter mode $\tilde{V}$ in which the number of the highest weight states is finite.

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